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Heat kernel, effective action and anomalies in noncommutative theories

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Abstract: Being motivated by physical applications (as the ϕ^4 model) we calculate the heat kernel coefficients for generalised laplacians on the Moyal plane containing both left and right multiplications. We found both star-local and star-nonlocal terms. By using these results we calculate the large mass and strong noncommutativity expansion of the effective action and of the vacuum energy. We also study the axial anomaly in the models with gauge fields acting on fermions from the left and from the right.

Keywords: Anomalies in Field and String Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Noncommutative field theories attract an ever increasing attention of the researches (see the reviews [1 − 3]). Various forms of noncommutativity arise from strings, gravity, deformation quantisation, and quantum Hall effect. Therefore, the possibility that our space-time is a noncommutative one should be taken seriously.

The heat kernel method [4 − 6] is one of the powerful instruments of modern quantum field theory. One considers an operator D which defines the spectrum of quantum fluctuations on a given background. For a bosonic theory, D is typically a second order operator of Laplace type. One defines the heat kernel (or, more precisely, the heat trace) for D as an L_2 trace

$$K(t, D) = \mathrm{Tr}_{L^2} (\exp(-tD)) \,, \tag{1.1}$$

where t is a spectral parameter. Usually, (see [4 − 6] for more precise statements) the heat trace is well defined for $t > 0$, and, as $t \to +0$, there is a full asymptotic expansion

$$K(t, D) \simeq \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(D) \,. \tag{1.2}$$

On manifolds without boundaries the odd-numbered coefficients vanish, $a_{2p+1} = 0$, and the even-numbered coefficients can be expressed via integrals over the manifold of local invariants constructed from the symbol of D.

In quantum field theory the coefficients in the expansion (1.2) (called also the heat kernel coefficients) define the one-loop divergences, anomalies, and various expansions of the effective action. In noncommutative theories the heat kernel expansion may be also used to find classical action through the so-called spectral action principle [7].

In the present paper we consider the case when the noncommutativity is defined through the Moyal star product. If the operator D contains left (or right) Moyal multiplications only, the corresponding heat kernel expansion was constructed in [8] on the torus, and in [9] on the plane. Curved space[1] modifications, conformal anomalies, and the Polyakov action were calculated in [12]. In the paper [13] it was noted that if D contains both right and left Moyal multiplications (and if the matrix $\theta^{\mu\nu}$, which defines the noncommutativity, is degenerate) the heat kernel expansion contains contributions which are not local even in the generalised noncommutative sense, so that some problems with renormalizability can arise. Since the presence of both left and right Moyal multiplications is required by many physical applications (to the noncommutative ϕ^4 theory, for example), we also consider such operators, but throughout the paper we assume that $\theta^{\mu\nu}$ is non-degenerate.[2] In the next section we propose a method for calculation of the heat kernel coefficients, and present explicit expressions for several leading terms. We find that there are two types of the coefficients. Leading coefficients are star-local (i.e., they are integrals of star-polynomials of the fields), but there are also rather non-standard non-local contributions, which are similar to the non-planar diagrams. We the help of the heat kernel expansion we construct the large mass expansion of the effective action and of the vacuum energy (section 3). This expansion appears to be a strong noncommutativity expansion (i.e., it is valid for large θ). Axial anomalies are considered in section 4, where we also find an anomaly-free model.

2. Heat kernel

The Moyal product of two functions f and g on $\mathbb{R}^n$ can be defined by the equation

$$f \star g = f(x) \exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu\right) g(x) . \tag{2.1}$$

θ is a constant antisymmetric matrix. In this form the star product has to be applied to plane waves and then extended to all (square integrable) functions by means of the Fourier series. The following properties of the Moyal star product will be used in this paper

$$\int d^n x\, f \star g = \int d^n x\, f \cdot g\,, \qquad \int d^n x\, f \star g \star h = \int d^n x\, h \star f \star g\,. \tag{2.2}$$

In this paper we consider the operators which can be represented in the form

$$D = -(\nabla_\mu^2 + E)\,, \tag{2.3}$$

where

$$\nabla_\mu = \partial_\mu + L(\lambda_\mu) + R(\rho_\mu)\,, \tag{2.4}$$
$$E = L(l_1) + R(r_1) + L(l_2) \circ R(r_2)\,. \tag{2.5}$$

L and R are operators of left and right Moyal multiplications,

$$L(l)\, f = l \star f\,, \qquad R(r)\, f = f \star r\,. \tag{2.6}$$

[1]From other heat kernel calculations on curved noncommutative spaces if the noncommutativity is realized in a different way we would like to mention the works [10, 11].

[2]The fact that non-degenracy of θ improves renormalisation was noted in [13]. This analysis was then extended to a position-dependent $\theta^{\mu\nu}$ in [14].

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The operator D we consider in this section acts on scalars. Additional matrix structure does not lead to much complications (see section 4). We use the symbol $\circ$ for the operator products to distinguish from the star product of functions. We shall, however, omit $\circ$ at some places if this cannot lead to a confusion. We suppose that all fields λ_μ, ρ_μ, $l_{1,2}$, $r_{1,2}$ fall off at the infinity faster than any power of a radial coordinate on $\mathbb{R}^n$. The mass dimension of λ_μ, ρ_μ, l_2 and r_2 is one. The fields l_1 and r_1 have mass dimension two.

Consider an L_2 scalar product on the space of functions

$$(f,g) = \int d^n x\, f^*(x) g(x)\,, \tag{2.7}$$

where f^* is a complex conjugate of f. With respect to this product formal adjoints of the multiplication operators read $R(r)^\dagger = R(r^*)$, $L(l)^\dagger = L(l^*)$.

There are two independent gauge symmetries of this problem (cf. [15] where this "double gauging" phenomenon was discovered in a different context):

$$D \to L(U_L^{-1}) \circ D \circ L(U_L)\,, \qquad \text{and} \qquad D \to R(U_R^{-1}) \circ D \circ R(U_R)\,. \tag{2.8}$$

The fields in the operator D transform according to the following rules

$$\begin{aligned}
\lambda_\mu &\to U_L^{-1} \star \partial_\mu U_L + U_L^{-1} \star \lambda_\mu \star U_L\,,\\
l_{1,2} &\to U_L^{-1} \star l_{1,2} \star U_L\,,\\
\rho_\mu &\to \partial_\mu U_R \star U_R^{-1} + U_R \star \rho_\mu \star U_R^{-1}\,,\\
r_{1,2} &\to U_R \star r_{1,2} \star U_R^{-1}\,.
\end{aligned} \tag{2.9}$$

The left fields λ_μ, l_1, l_2 (respectively, the right fields ρ_μ, r_1, r_2) are invariant under the transformations parametrised by U_R (respectively, by U_L). By representing $U_{L,R} = e^{w_{L,R}}$ and restricting (2.9) to the linear order in $w_{L,R}$ one obtains an infinitesimal version of the gauge transformations

$$\begin{aligned}
\delta\lambda_\mu &= \partial_\mu w_L + [\lambda_\mu, w_L]\,,\\
\delta l_{1,2} &= [l_{1,2}, w_L],\\
\delta\rho_\mu &= \partial_\mu w_R + [w_R, \rho_\mu],\\
\delta r_{1,2} &= [w_R, r_{1,2}]\,.
\end{aligned} \tag{2.10}$$

In the exponential and in the commutators all products are the Moyal star products.

The gauge fields, λ_μ and ρ_μ, and the gauge parameters $w_{L,R}$ are typically pure imaginary, so that $U_{L,R}^{-1} = U_{L,R}^*$. There are two different field strengths

$$\nabla_\mu \circ \nabla_\nu - \nabla_\nu \circ \nabla_\mu = L(\Omega^L_{\mu\nu}) + R(\Omega^R_{\mu\nu})\,, \tag{2.11}$$

where

$$\begin{aligned}
\Omega^L_{\mu\nu} &= \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu + [\lambda_\mu, \lambda_\nu]\\
\Omega^R_{\mu\nu} &= \partial_\mu \rho_\nu - \partial_\nu \rho_\mu + [\rho_\nu, \rho_\mu]\,.
\end{aligned} \tag{2.12}$$

We also introduce two covariant derivatives

$$\nabla^L_\mu = \partial_\mu + [\lambda_\mu, \,\cdot\,] , \qquad \nabla^R_\mu = \partial_\mu + [\,\cdot\,, \rho_\mu] . \tag{2.13}$$

Let us now evaluate the asymptotic expansion of the trace of heat kernel

$$K(t, D) = \mathrm{Tr}_{L^2} \left(e^{-tD} - e^{-tD_0} \right) . \tag{2.14}$$

In order to remove a trivial volume divergence of the trace we have subtracted the heat kernel of the "free" operator $D_0 = -\partial^2_\mu$. Alternatively, this volume divergence can be regularised by introducing a smearing function under the trace (see section 4). From now on we suppose that the matrix $\theta^{\mu\nu}$ is non-degenerate. Consequently, the dimension n must be even.

To evaluate the L_2 trace we sandwich the exponentials between plane waves e^{ikx}, integrate over x (to produce diagonal matrix elements of the heat operator), and finally integrate over k (to calculate the trace).

$$K(t, D) = \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-ikx} \left(e^{-tD} - e^{-tD_0} \right) e^{ikx} . \tag{2.15}$$

In order to evaluate the asymptotic expansion of (2.15) at $t \to +0$ one has to extract the factor e^{-tk^2}.

$$\begin{aligned} K(t, D) = & \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2} \times \\ & \times \langle \exp \left(t \left((\nabla - ik)^2 + 2ik^\mu (\nabla_\mu - ik_\mu) + E \right) \right) - 1 \rangle_k , \end{aligned} \tag{2.16}$$

where we defined

$$\langle F \rangle_k \equiv e^{-ikx} \star F e^{ikx} \tag{2.17}$$

for any operator F. Next one has to expand the exponential in (2.16) in a power series in E and $(\nabla - ik)$. Only a finite number of terms in this expansion contribute to any finite order of t in the $t \to +0$ asymptotic expansion of the heat kernel.

To illustrate the method let us consider the terms which are obtained by expanding the exponential in (2.16) up to the quadratic order in E.

$$K(t, D)_{E^2} = \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2} \langle tE + \frac{t^2}{2} E^2 \rangle_k . \tag{2.18}$$

Equation (2.5) yields

$$\begin{aligned} E^2 = \; & R(r_1 \star r_1) + L(l_1 \star l_1) + R(r_2 \star r_2) \circ L(l_2 \star l_2) + 2R(r_1) \circ L(l_1) + \\ & + R(\{r_1, r_2\}) \circ L(l_2) + R(r_2) \circ L(\{l_1, l_2\}) . \end{aligned} \tag{2.19}$$

The terms containing left or right multiplications only can be dealt with easily by using the identities

$$\int d^n x \langle R(r) \rangle_k = \int d^n x \, r(x) , \qquad \int d^n x \langle L(l) \rangle_k = \int d^n x \, l(x) \tag{2.20}$$

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and the integral

$$\int \frac{d^n k}{(2\pi)^n} e^{-tk^2} = (4\pi t)^{-n/2} \,. \tag{2.21}$$

The terms containing both left and right multiplications are somewhat more difficult. Consider a typical contribution of that type

$$T(l,r) = \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2} \langle L(l) \circ R(r) \rangle_k \tag{2.22}$$

with some functions $r(x)$ and $l(x)$. Let us expand $r(x)$ and $l(x)$ in the Fourier integrals

$$\begin{aligned} r(x) &= \frac{1}{(2\pi)^{n/2}} \int d^n q \, r(q) e^{iqx} \,, \\ l(x) &= \frac{1}{(2\pi)^{n/2}} \int d^n q' \, l(q') e^{iq'x} \,. \end{aligned} \tag{2.23}$$

Then

$$\langle L(l) \circ R(r) \rangle_k = \frac{1}{(2\pi)^n} \int d^n q \, d^n q' \, r(q) l(q') e^{i(q+q')x} e^{\frac{i}{2} k \wedge (q-q')} e^{-\frac{i}{2}(q'-k)\wedge(q+k)} \,, \tag{2.24}$$

where

$$k \wedge q \equiv \theta^{\mu\nu} k_\mu q_\nu \,. \tag{2.25}$$

Next we substitute (2.24) in (2.22) and integrate over x and q' to obtain

$$T(l,r) = \int \frac{d^n k \, d^n q}{(2\pi)^n} e^{-tk^2} l(-q) \, r(q) e^{-ik\wedge q} \,. \tag{2.26}$$

To calculate the integral over k we complete the square in the exponential

$$-tk^2 - ik \wedge q = -t \left(k_\mu + \frac{i}{2t} \theta_\mu{}^\nu q_\nu \right)^2 - \frac{1}{4t} \theta^{\mu\mu'} \theta^\nu{}_{\mu'} q_\mu q_\nu \,. \tag{2.27}$$

Then equation (2.26) becomes

$$T(l,r) = \int \frac{d^n q}{(4\pi t)^{n/2}} l(-q) r(q) \exp\left(-\frac{1}{4t} \theta^{\mu\mu'} \theta^\nu{}_{\mu'} q_\mu q_\nu \right) . \tag{2.28}$$

To evaluate the asymptotic behaviour of (2.28) at $t \to +0$ one has to expand $l(-q)r(q)$ in Taylor series and then perform the gaussian integral over q. First two terms of this expansion read

$$T(l,r) = (\det\theta)^{-1} \left[l(0)r(0) + t \left(\theta\theta^T\right)^{-1}_{\mu\nu} \frac{\partial^2}{\partial q_\mu \partial q_\nu} \left(l(-q) r(q) \right)_{q=0} + \mathcal{O}(t^2) \right] . \tag{2.29}$$

Next one returns to the coordinate representation by using the following formulae

$$l(0)r(0) = \frac{1}{(2\pi)^n} \int d^n x \, l(x) \int d^n y \, r(y) \,, \tag{2.30}$$

$$\begin{aligned} \frac{\partial^2}{\partial q_\mu \partial q_\nu} \left(l(-q) r(q) \right)_{q=0} = \frac{1}{(2\pi)^n} \Big[&- \int d^n x \, x^\mu x^\nu l(x) \int d^n y \, r(y) - \\ &- \int d^n x l(x) \int d^n y \, y^\mu y^\nu r(y) + \int d^n x x^\mu l(x) \int d^n y y^\nu r(y) + \\ &+ \int d^n x \, x^\nu l(x) \int d^n y \, y^\mu r(y) \Big] . \end{aligned} \tag{2.31}$$

Note that both expressions in (2.30) and (2.31) are invariant under constant shifts of the coordinates x and y, i.e. they do not depend on the point where we put an origin of our coordinate system. The integrals in (2.31) are not gauge invariant. We shall return to this issue later.

By simply collecting the formulae given above one can easily find the contributions of E the heat kernel coefficients up to the E^2 order (cf. (2.18)). All other terms can be treated in a similar way. In general case one can prove the following statements.

(i) There is a power law asymptotic expansion of the heat kernel (1.2). All odd-numbered coefficients a_k with $k = 2j+1$ vanish, while the even-numbered coefficients with $k = 2j$ are given by the formula:

$$a_k(D) = a_k^L(D) + a_k^R(D) + a_k^{\rm mix}(D) \,. \tag{2.32}$$

(ii) The coefficients $a_k^L(D)$ (respectively, $a_k^R(D)$) depend on the left (respectively, right) fields only. They are represented by integral over $\mathbb{R}^n$ of gauge-invariant star polynomials. These coefficients have been calculated earlier in [8] (on the noncommutative torus) and in [9] (on the noncommutative plane). First several coefficients read

$$a_2^L = (4\pi)^{-n/2} \int d^n x l_1(x) \,, \tag{2.33}$$

$$a_4^L = (4\pi)^{-n/2} \frac{1}{12} \int d^n x \left(6 l_1 \star l_1 + \Omega^{L\mu\nu} \star \Omega^L_{\mu\nu} \right) \tag{2.34}$$

$$\begin{aligned} a_6^L = (4\pi)^{-n/2} \frac{1}{360} \int d^n x \Big(& 60 l_1 \star l_1 \star l_1 + 30 l_1 \star \nabla^{L\mu} \nabla^L_\mu l_1 + \\ & + 30 l_1 \star \Omega^L_{\mu\nu} \star \Omega^{L\mu\nu} - 4 (\nabla^L_\sigma \Omega^L_{\mu\nu}) \star (\nabla^{L\sigma} \Omega^{L\mu\nu}) + \\ & + 2 (\nabla^{L\nu} \Omega^L_{\mu\nu}) \star (\nabla^L_\sigma \Omega^{L\mu\sigma}) - 12 \Omega^L_{\mu\nu} \star \Omega^{L\nu}{}_\sigma \star \Omega^{L\sigma\mu} \Big) \,, \end{aligned} \tag{2.35}$$

The coefficients a_k^R are obtained from a_k^L by replacing $\{l_1, \nabla^L, \Omega^L\}$ with $\{r_1, \nabla^R, \Omega^R\}$ and the reversing the order of all multipliers under the integral[3].

(iii) The mixed coefficients $a_k^{\rm mix}(D)$ vanish for $k \le n$. The first non-zero coefficient is

$$a_{n+2}^{\rm mix} = \frac{(\det \theta)^{-1}}{(2\pi)^n} \left(\int d^n x \, l_2(x) \int d^n y \, r_2(y) + 2 \int d^n x \, \lambda^\mu(x) \int d^n y \, \rho_\mu(y) \right) . \tag{2.36}$$

Let $\lambda_\mu = \rho_\mu = 0$. Then the next coefficient reads

$$\begin{aligned} a_{n+4}^{\rm mix} = \frac{(\det \theta)^{-1}}{(2\pi)^n} \Big[& \frac{1}{2} \int d^n x \, l_2^2 \int d^n y \, r_2^2 + \int d^n x \, l_1 \int d^n y \, r_1 + \\ & + \int d^n x \, l_2 \int d^n y \, r_1 \star r_2 + \int d^n x \, l_1 \star l_2 \int d^n y \, r_2 + \\ & + (\theta \theta^T)^{-1}_{\mu\nu} \Big(- \int d^n x \, x^\mu \star x^\nu \star l_2 \int d^n y \, r_2 - \end{aligned}$$

[3]This inversion rule follows from the identity $R(r) \circ R(r') = R(r' \star r)$. The inversion is partially taken care of by the definitions of the covariant derivatives (2.13) and the field strengths (2.12).

$$
\begin{aligned}
&- \int d^n x\, l_2 \int d^n y\, y^\mu \star y^\nu \star r_2 + \\
&+ 2 \int d^n x\, x^\mu l_2 \int d^n y\, y^\nu \star r_2) \Big] .
\end{aligned} \tag{2.37}
$$

The coefficients (2.33)–(2.36) are gauge invariant. The expression (2.37) is not gauge invariant since we assumed $\lambda_\mu = \rho_\mu = 0$. If the gauge fields are non-zero, the coordinates x^μ and y^μ which appear under the integrals are replaced by the expressions

$$
X^\mu = x^\mu + i\theta^{\mu\sigma}\lambda_\sigma \qquad \text{and} \qquad Y^\mu = y^\mu - i\theta^{\mu\sigma}\rho_\sigma \tag{2.38}
$$

respectively. These shifted coordinates are gauge covariant, $\delta X^\mu = [X^\mu, w_L]$, $\delta Y^\mu = [w_R, Y^\mu]$. Therefore, the gauge invariance of the heat kernel expansion is restored.

Note, that if $\theta^{\mu\nu}$ is degenerate, mixed contributions to the heat kernel coefficients may appear earlier than in a_{n+2}, which may affect renormalization [13]. In the quantum field theory language, mixed coefficients correspond to non-planar diagrams. We shall discuss these points below.

3. Effective action and vacuum energy

Let us first recall some basic facts regarding the one-loop effective action in quantum field theory. Consider a scalar field ϕ described by a classical action $S(\phi)$. In the background field formalism one splits $\phi = \varphi + \delta\phi$, where φ is a background field. The field $\delta\phi$ describes quantum fluctuations. Then one expands $S(\phi)$ about the background. The first term, $S(\varphi)$, simply gives the classical approximation to the effective action. The second term, which is proportional to the first derivative of $S(\varphi)$ is cancelled by external sources. The quadratic term can be rewritten as

$$
S^{(2)} = \int d^n x\, (\delta\phi)(D + m^2)(\delta\phi)\,, \tag{3.1}
$$

where D is an operator which depends on the background field φ. We have separated explicitly the mass term m^2. The functional integration of $\exp(-S^{(2)})$ defines the one loop effective action

$$
W = -\ln \int \mathcal{D}(\delta\phi) e^{-S^{(2)}} = \frac{1}{2} \ln\det(D + m^2)\,. \tag{3.2}
$$

This expression is, of course, divergent and has to be regularised. We use the zeta function regularization [16, 17]. The zeta function is defined by the equation

$$
\zeta(s, D + m^2) = \mathrm{Tr}_{L^2}\left((D + m^2)^{-s} - (D_0 + m^2)^{-s} \right), \tag{3.3}
$$

where s is a complex spectral parameter. Again, as in (2.14), we subtracted the zeta function of the free operator D_0 in order to remove a trivial volume divergence.

The regularised effective action reads

$$
W_s = -\frac{1}{2} \tilde{\mu}^{2s} \Gamma(s) \zeta(s, D + m^2)\,. \tag{3.4}
$$

The regularization is removed in the limit $s \to 0$. $\tilde{\mu}$ is a constant of the dimension of mass introduced to keep proper dimension of the effective action. The heat kernel and the zeta function are related by a Mellin transformation. One can rewrite (3.4) as

$$W_s = -\frac{1}{2}\tilde{\mu}^{2s} \int_0^\infty \frac{dt}{t^{1-s}} K(t,D) e^{-tm^2} . \tag{3.5}$$

If m^2 is large enough the integral over t is convergent at the upper limit. There are, however, divergences at the lower limit which are defined by the heat kernel expansion. By substituting (1.2) in (3.5) one obtains

$$W_s \simeq -\frac{1}{2}\left(\frac{\tilde{\mu}}{m}\right)^{2s} \sum_k \Gamma\left(s + \frac{k-n}{2}\right) m^{n-k} a_k(D) . \tag{3.6}$$

Let us recall, that the coefficient $a_0(D)$ vanishes because of the subtraction of the volume term, the odd-numbered coefficients a_{2j+1} vanish on manifolds without boundaries in both commutative and non-commutative cases. Consequently, the summation in (3.6) runs over even positive integers, $k = 2, 4, \ldots$ We have assumed that n is even. The gamma functions in (3.6) have poles at $s = 0$ for $k \le n$, and the corresponding terms in the sum define one-loop divergences. According to the results obtained in the previous section the coefficients $a_k(D)$ with $k \le n$ are integrals of star-polynomials constructed from fields and their derivatives, i.e. the divergences have a structure typical to the classical actions. One may hope therefore, that the divergences may be absorbed in a redefinition of the coupling constants. Particular normalisation conditions depend, of course, on the model in question. Since the divergent terms are always proportional to non-negative powers of the mass, the scheme based on the subtraction of leading terms in the $m \to \infty$ asymptotics [18] should work anyway, although its physical meaning is not always clear.

With the help of (3.6) one can also evaluate the large mass expansion of the effective action. First several terms in (3.6) are divergent. Therefore, as discussed above, the corresponding terms in the large mass expansion are defined by the renormalization of the classical action on a given background. The terms with $k > n$ are non-divergent. They represent "genuine" quantum corrections to the effective action (since corresponding structures may be absent in the classical action). In these terms we can put $s = 0$ thus obtaining the following expression

$$W^{[1/m]} = -\frac{1}{2}\sum_{p=1}^{\infty} m^{-2p}(p-1)!\, a_{2p+n}(D) . \tag{3.7}$$

In the commutative case, all terms of these expansion are local since the heat kernel coefficients are local. This property is clearly lost in the noncommutative case.

As an example, let us consider a real scalar field ϕ in four dimensions with the classical action[4]

$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{24}\phi \star \phi \star \phi \star \phi\right) , \tag{3.8}$$

[4]Sometimes the ϕ^4 action on noncommutative plane is modified by an external oscillator potential [19].

where g is a coupling constant. By expanding this action around a background field φ and keeping the terms which are quadratic in the fluctuations $\delta\phi$ only one arrives at (3.1) with (cf. [13])

$$D = -\partial_\mu^2 + \frac{g}{6}\left[R(\varphi\star\varphi) + L(\varphi\star\varphi) + L(\varphi)\circ R(\varphi)\right]. \tag{3.9}$$

This operator corresponds to the following choice in (2.3)–(2.5)

$$\lambda_\mu = \rho_\mu = 0\,, \quad l_1 = r_1 = -\frac{g}{6}\varphi\star\varphi\,, \quad l_2 = -r_2 = \sqrt{\frac{g}{6}}\varphi\,. \tag{3.10}$$

The terms with $k = 2, 4$ in (3.6) are divergent. Near $s = 0$ they read

$$W^{\rm div} = -\frac{1}{2}\left[\left(-\frac{1}{s} + \gamma_E - 1 - \ln\left(\frac{\tilde{\mu}^2}{m^2}\right)\right)m^2 a_2 + \left(\frac{1}{s} - \gamma_E + \ln\left(\frac{\tilde{\mu}^2}{m^2}\right)\right)a_4\right] + \mathcal{O}(s)\,, \tag{3.11}$$

where

$$\begin{aligned} a_2 &= -\frac{g}{24\pi^2}\int d^4x\,\varphi^2\,, \\ a_4 &= \frac{1}{16\pi^2}\frac{g^2}{36}\int d^4x\,\varphi\star\varphi\star\varphi\star\varphi\,. \end{aligned} \tag{3.12}$$

We see now that the divergences in (3.11) can indeed be absorbed in redefinitions of the mass and of the coupling constant in the classical action (3.8) (which is very well known).

By substituting (3.10) in (2.35)–(2.37) and then in (3.7) one obtains

$$\begin{aligned} W^{[1/m]} = & -\frac{1}{32\pi^2}\frac{g^2}{648m^2}\int d^4x\left(-g\varphi_\star^6 + 3\varphi_\star^2\partial_\mu^2\varphi_\star^2\right) + \\ & + \frac{(\det\theta)^{-1}}{(2\pi)^4 m^2}\left[\frac{g}{12}\left(\int d^4x\,\varphi\right)^2 - \frac{g^2}{48m^2}\left(\int d^4x\,\varphi^2\right)^2 - \right. \\ & \quad - \frac{g^2}{36m^2}\int d^4x\;\varphi\int d^4y\,\varphi_\star^3 - \frac{g}{6m^2}(\theta\theta^T)^{-1}_{\mu\nu}\times \\ & \quad \left.\times\left(\int d^4x x^\mu\star x^\nu\star\varphi\int d^4y\varphi - \int d^4x\,x^\mu\star\varphi\int d^4yy^\nu\star\varphi\right)\right] + \cdots\,, \end{aligned} \tag{3.13}$$

where $\varphi_\star^k$ is the kth star-power of φ. E.g., $\varphi_\star^3 \equiv \varphi\star\varphi\star\varphi$.

As in the commutative case, this expansion is valid if the fields and their derivatives are small compare to the mass. In the noncommutative case $\theta^{\mu\nu}$ appears in the denominator, so one should also assume that θ^{-1} is small in a natural scale defined by the mass. Therefore, we have constructed a *strong coupling expansion* with respect to the noncommutativity parameter. Consequently, there is no smooth transition to the case of a degenerate $\theta^{\mu\nu}$.

With the same technical tools we can also evaluate the vacuum (Casimir) energies[5] of static $n+1$ dimensional systems. We still assume that n is even, that the noncommutativity is confined to n spatial dimensions, and that θ is a non-degenerate $n\times n$ matrix. In the zeta-function regularization the ground state energy is defined as

$$\mathcal{E}_s = \frac{1}{2}\mu^{2s}\sum_p \varepsilon_p^{1-2s}\,, \tag{3.14}$$

[5] For a recent review on the Casimir energy see [20]. Here we follow ref. [21].

where ε_p are eigenfrequencies of elementary excitations defined as square root of eigenvalues of the hamiltonian

$$H = D + m^2 \,. \tag{3.15}$$

Formally taking $s = 0$ in (3.14) yields just a sum of zero point energies of elementary oscillators. We rewrite eq. (3.14) through the zeta function of H,

$$\mathcal{E}_s = \frac{1}{2}\mu^{2s}\zeta\left(s - \frac{1}{2}, H\right). \tag{3.16}$$

The zeta function can be expressed in terms of the heat kernel

$$\zeta\left(s - \frac{1}{2}, H\right) = \frac{1}{\Gamma\left(s - \frac{1}{2}\right)}\int_0^\infty dt\, t^{s-3/2} K(t, D) e^{-tm^2} \,. \tag{3.17}$$

We assume that the operator D is as in section 2, so that the expansion (1.2) with integer powers of t exists. Then (3.17) is finite at $s = 0$ and a large mass expansion of the vacuum energy exists without any infinite renormalization of the couplings[6]

$$\mathcal{E}^{[1/m]} = -\frac{1}{4\sqrt{\pi}}\sum_{p=1}^{\infty} a_{2p}(D)\, m^{n+1-2p}\Gamma\left(p - \frac{n+1}{2}\right). \tag{3.18}$$

As an example we consider a model in 2+1 dimensions ($n = 2$). In this case, $\theta^{ij} = \Theta\epsilon^{ij}$, where θ is a constant and ϵ^{ij} is the Levi-Civita tensor. $i, j = 1, 2$ are the space-like indices. Again we consider the ϕ^4 theory, so that the operator is given by eq. (3.9) with μ, ν replaced by the two-dimensional indices i, j.

$$\mathcal{E}^{[1/m]} = -\frac{gm}{24\pi}\int d^2x\, \varphi^2 - \frac{g^2}{576\pi m}\int d^2x\, \varphi_\star^4 + \frac{g}{96\pi^2 m}\Theta^{-2}\left[\int d^2x\, \varphi\right]^2 + \cdots \,. \tag{3.19}$$

This expansion is valid if φ, its derivatives, and $1/\Theta$ are small compared to the mass. It would be interesting to apply this expansion to quantum corrections to noncommutative solitons. In the commutative case the heat kernel methods give rapidly convergent series for the mass shift even if there is no explicitly small parameter in the model [22]. For supersymmetric solitons the heat kernel methods can even give exact results for the mass shift [23], but they require supersymmetric boundary conditions on quantum fluctuations.

The very appearance of mixed terms in the heat kernel expansion and of the corresponding terms in the effective action is a consequence of qualitatively different behaviour of planar and non-planar diagrams in noncommutative theories and of the UV/IR (ultraviolet/infrared) mixing phenomenon [24 – 26]. There are some similarities between general structure of the mixed heat kernel coefficients and that of non-planar diagrams. The heat kernel expansion provides a systematic (and relatively simple) way to construct a large m and strong noncommutativity expansion of the one-loop effective action.[7]

It is essential here that the space has a trivial topology. For a non-trivial topology many interesting effects may occur [28].

[6] This property always holds in commutative zeta-regularised theories in odd dimensions. In the noncommutative case it is essential that θ is non-degenerate. Note that in both commutative and noncommutative cases a finite renormalization of couplings may occur.

[7] A diagrammatic approach to the effective action in the noncommuative ϕ^4 theory can be found in [27].

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4. Localised heat kernel and anomalies

Let us consider a classical action for the Dirac spinors on the Moyal plane

$$S_\psi = \int d^n x\, \bar{\psi} \star \not{D} \psi \,, \tag{4.1}$$

where, in the euclidean space, $\bar{\psi} = \psi^\dagger$. We choose the Dirac operator in the form

$$\not{D} = i\gamma^\mu \left(\partial_\mu + iL(V_\mu^L) + iR(V_\mu^R) + \gamma_5 L(A_\mu^L) + \gamma_5 R(A_\mu^R) \right) . \tag{4.2}$$

The Dirac gamma matrices satisfy the Clifford relation $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$. Independently of the dimension n the chirality matrix is denoted by γ_5, and $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$, $\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = 1$.

The classical action (4.1) is invariant under a two parameter family of infinitesimal gauge transformations

$$\begin{aligned}
\delta_w \psi &= iw_L \star \psi + i\psi \star w_R \,, \qquad \delta_w \bar{\psi} = -i\bar{\psi} \star w_L - iw_R \star \bar{\psi} \,, \\
\delta_w V_\mu^L &= -\partial_\mu w_L - i[V_\mu^L, w_L] \,, \\
\delta_w V_\mu^R &= -\partial_\mu w_R + i[V_\mu^R, w_R] \,, \\
\delta_w A_\mu^L &= -i[A_\mu^L, w_L] \,, \qquad \delta_w A_\mu^R = i[A_\mu^R, w_R]
\end{aligned} \tag{4.3}$$

and under a two-parameter family of infinitesimal axial gauge transformations

$$\begin{aligned}
\delta_\sigma \psi &= \sigma_L \star \gamma_5 \psi + \gamma_5 \psi \star \sigma_R \,, \qquad \delta_\sigma \bar{\psi} = \sigma_R \star \bar{\psi}\gamma_5 + \bar{\psi}\gamma_5 \star \sigma_L \,, \\
\delta_\sigma V_\mu^L &= i[A_\mu^L, \sigma_L] \,, \qquad \delta_\sigma V_\mu^R = -i[A_\mu^R, \sigma_R] \,, \\
\delta_\sigma A_\mu^L &= -\partial_\mu \sigma_L - i[V_\mu^L, \sigma_L] \,, \\
\delta_\sigma A_\mu^R &= -\partial_\mu \sigma_R + i[V_\mu^R, \sigma_R] \,.
\end{aligned} \tag{4.4}$$

Here we assume that $V_\mu^{L,R}$, $A_\mu^{L,R}$, $w_{L,R}$ and $\sigma_{L,R}$ are real. This explains some extra factors of i as compared to section 2.

Now we like to define which of the symmetries (4.3) and (4.4) are anomalous and calculate corresponding anomalies. After integrating out the fermions one arrives at the effective action

$$W^{[f]} = -\ln \det \not{D} = -\frac{1}{2} \ln \det \not{D}^2 \,. \tag{4.5}$$

Again we use the zeta-function methods to regularise the determinant (4.5). We write the regularised effective action as (cf. (3.4))

$$W_s^{[f]} = \frac{1}{2} \tilde{\mu}^2 \Gamma(s) \zeta(s, \not{D}^2) \,. \tag{4.6}$$

Up to a certain point the calculation of the anomaly in the zeta-function regularization goes precisely the same way as in the commutative case [29 − 31]. After subtracting the pole at $s = 0$ the effective action becomes

$$W^{[f]} = \frac{1}{2} \zeta(0, \not{D}^2)' + \frac{1}{2} \ln(\mu^2) \zeta(0, \not{D}^2) \,, \tag{4.7}$$

where prime denotes differentiation with respect to s. The renormalization ambiguity resides now in the constant μ^2, which has to be determined through a normalisation condition.

The variation of the zeta function induced by any variation of the external fields $V_\mu^{L,R}$, $A_\mu^{L,R}$ in $\not{D}$ is

$$\delta\zeta(s,\not{D}^2) = -2s\mathrm{Tr}\left((\delta\not{D})\not{D}\not{D}^{2(-s-1)}\right). \tag{4.8}$$

The variation of the Dirac operator under the gauge transformations (4.3) reads

$$\delta_w\not{D} = -i[\not{D}, L(w_L) + R(w_R)]. \tag{4.9}$$

The substitution of (4.9) in (4.8) yields $\delta_w\zeta(s,\not{D}^2) = 0$ (where we used cyclic symmetry of the trace). Consequently, $\delta_w W^{[f]} = 0$. We conclude that the zeta-function regularization preserves gauge symmetries.

For the axial gauge transformations (4.4) we have

$$\delta_\sigma\not{D} = -\{\not{D}, \gamma_5(L(\sigma_L) + R(\sigma_R))\} \tag{4.10}$$

so that the variation of the zeta function reads

$$\delta_\sigma\zeta(s,\not{D}^2) = 4s\mathrm{Tr}\left(\gamma_5(L(\sigma_L) + R(\sigma_R))\not{D}^{-2s}\right) \equiv 4s\zeta\left(\gamma_5(L(\sigma_L) + R(\sigma_R)), s, \not{D}^2\right), \tag{4.11}$$

where we defined a smeared (or localised) zeta function

$$\zeta(F,s,D) \equiv \mathrm{Tr}\left(FD^{-s}\right). \tag{4.12}$$

In a similar way also a smeared heat kernel can be defined

$$K(F,t,D) \equiv \mathrm{Tr}\left(Fe^{-tD}\right). \tag{4.13}$$

We assume that F is a zeroth order operator (i.e., it contains multiplication, but no explicit partial derivatives). As we will see below, there is an asymptotic expansion as in the unsmeared case

$$K(F,t,D) \simeq \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(F,D). \tag{4.14}$$

There is no need to subtract from the smeared heat kernel the contribution of the "free" operator (cf. (2.14)) since the volume divergences can be removed if the smearing functions σ_L and σ_R fall off sufficiently fast at the infinity. Moreover, in the present case $\mathrm{Tr}\left(\gamma_5(L(\sigma_L) + R(\sigma_R))e^{-tD_0}\right) = 0$ because of the presence of γ_5 under the trace.

The heat kernel and the zeta function are related through the Mellin transform. In particular,

$$a_k(F,D) = \mathrm{Res}_{s=(n-k)/2}(\Gamma(s)\zeta(F,s,D)) \tag{4.15}$$

and $a_n = \zeta(F,0,D)$.

Finally one arrives at the following expression for the axial anomaly

$$\mathcal{A}_\sigma \equiv \delta_\sigma W^{[f]} = 2a_n\left(\gamma_5(L(\sigma_L) + R(\sigma_R))\not{D}^2\right). \tag{4.16}$$

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The Fujikawa approach [32] and the finite-mode regularization method [33] give a similar expression for the anomaly (up to some peculiarities arising from the presence of a dimensional regularization parameter in that schemes).

The square of the Dirac operator can be represented in a form similar to (2.3):

$$\not{D}^2 = -((\partial_\mu + \omega_\mu)^2 + E)\,, \tag{4.17}$$

where

$$\begin{aligned} \omega_\mu &= iL(V^L_\mu) + iR(V^R_\mu) + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\gamma_5 \left(L(A^L_\nu) + R(A^R_\nu)\right), \qquad (4.18)\\ E &= \frac{i}{4}[\gamma^\mu, \gamma^\nu]\left(L(V^L_{\mu\nu}) + R(V^R_{\mu\nu})\right) + \gamma_5\left(L(\nabla^{\mu L}A^L_\mu) + R(\nabla^{\mu R}A^R_\mu)\right) + \\ &+ (n-2)\left(L(A^{\mu L}\star A^L_\mu) + R(A^{\mu R}\star A^R_\mu) + 2L(A^L_\mu)\circ R(A^{\mu R})\right) + \\ &+ \frac{1}{4}(n-3)[\gamma^\mu, \gamma^\nu]\left(L([A^L_\mu, A^L_\nu]) + R([A^R_\nu, A^R_\mu])\right). \qquad (4.19) \end{aligned}$$

We have defined

$$\begin{aligned} V^L_{\mu\nu} &= \partial_\mu V^L_\nu - \partial_\nu V^L_\mu + i\left[V^L_\mu, V^L_\nu\right], & V^R_{\mu\nu} &= \partial_\mu V^R_\nu - \partial_\nu V^R_\mu + i\left[V^R_\nu, V^R_\mu\right], \\ \nabla^L_\mu A^L_\nu &= \partial_\mu A^L_\nu + i\left[V^L_\mu, A^L_\nu\right], & \nabla^R_\mu A^R_\nu &= \partial_\mu A^R_\nu - i\left[V^R_\mu, A^R_\nu\right]. \end{aligned} \tag{4.20}$$

Now we are ready to calculate the heat kernel coefficient in (4.16). There are two differences to the case considered in section 2: both ω_μ and E are matrix-valued, and the heat kernel is smeared with a zeroth order operator. An extension to a matrix valued D goes almost without an effort. All steps go through, but one has to replace λ_μ, ρ_μ, $l_{1,3}$ and $r_{1,2}$ by the matrices which follow from (4.18) and (4.19). Because of the matrix structure, the terms with right multiplications do not commute any more with the terms with left multiplications, but this effect is important in "mixed" coefficients only. The smearing operator will appear linearly in all expressions, and one should not forget to take a trace over the spinor indices. For example, the E-terms in the heat kernel (cf. (2.18 in the unsmeared case) up to the quadratic order in E read

$$K(F,t,D)_{E^2} = \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2}\,\mathrm{tr}\left\langle F\left(1 + tE + \frac{t^2}{2}E^2\right)\right\rangle_k, \tag{4.21}$$

where tr is the γ-matrix trace. For $n=2$ and $n=4$ one can calculate the anomaly by expanding the exponents, as we have outlines in section 2 above. This way is the easiest one from the conceptual point of view, but it is also rather lengthy. More experienced readers can choose a different way based on functorial properties of the heat kernel (some useful tools can be found in [34]). First one proves that the coefficient $a_n(F,D)$ does not contain mixed contributions as in the unsmeared case, then one classifies the invariants of proper dimension which may appear in $a_n(F,D)$. The numerical coefficients in front of these invariants are then defined by comparing to the results for noncommutative heat kernel [8] and for the commutative axial anomaly [33]. Both methods, of course, give identical results.

In two dimensions the anomaly reads

$$\mathcal{A}_\sigma = -\frac{i}{\pi}\int d^2x\Big[\sigma_L\star\left(\frac{1}{2}\epsilon^{\mu\nu}(iV^L_{\mu\nu}-[A^L_\mu,A^L_\nu])+i\nabla^L_\mu A^{L\mu}\right)+ \\ +\sigma_R\star\left(\frac{1}{2}\epsilon^{\mu\nu}(iV^R_{\mu\nu}+[A^R_\mu,A^R_\nu])+i\nabla^R_\mu A^{R\mu}\right)\Big]. \tag{4.22}$$

This result is consistent with earlier calculations [35, 36] performed in the models either without V^R_μ [35], or when V^R_μ and V^L_μ act on different field components [36].

In four dimensions, it is useful to split the anomaly in four contributions

$$\mathcal{A}_\sigma = \mathcal{A}^+_{\sigma_L}+\mathcal{A}^-_{\sigma_L}+\mathcal{A}^+_{\sigma_R}+\mathcal{A}^-_{\sigma_R}\,, \tag{4.23}$$

$$\mathcal{A}^+_{\sigma_L} = \frac{-i}{24\pi^2}\int d^4x\sigma_L\star\Big(-4[\nabla^L_\mu V^{L\mu\nu},A^L_\nu]+2[\nabla^L_\mu A^L_\nu,V^{L\mu\nu}]+2i\nabla^L_\mu\nabla^{L\mu}\nabla^L_\nu A^{L\nu}+ \\ +4i\{\{\nabla^L_\mu A^L_\nu,A^{L\nu}\},A^{L\mu}\}+2i\{\nabla^L_\mu A^{L\mu},A^L_\nu\star A^{L\nu}\}+ \\ +4iA^L_\mu\star(\nabla^L_\nu A^{L\nu})\star A^{L\mu}+i[[A^L_\mu,A^L_\nu],A^{L\mu\nu}]\Big) \tag{4.24}$$

$$\mathcal{A}^-_{\sigma_L} = \frac{-i}{48\pi^2}\int d^4x\sigma_L\star\epsilon^{\mu\nu\rho\sigma}\Big(3iV^L_{\mu\nu}\star V^L_{\rho\sigma}-iA^L_{\mu\nu}\star A^L_{\rho\sigma} \\ -2(V^L_{\mu\nu}\star A^L_\rho\star A^L_\sigma+A^L_\mu\star A^L_\nu\star V^L_{\rho\sigma})-8A^L_\mu\star V^L_{\nu\rho}\star A^L_\sigma+ \\ +4iA^L_\mu\star A^L_\nu\star A^L_\rho\star A^L_\sigma\Big)\,, \tag{4.25}$$

where $A^L_{\mu\nu}=\nabla^L_\mu A^L_\nu-\nabla^L_\nu A^L_\mu$.

Due to the identity (cf. [33] for the commutative case)

$$\mathcal{A}^+_{\sigma_L} = \frac{1}{48\pi^2}\,\delta_{\sigma_L}\int d^4x\Big(-V^L_{\mu\nu}\star V^{L\mu\nu}-2(\nabla^L_\mu A^{L\mu})\star(\nabla^L_\nu A^{L\nu})-4iA^L_\mu\star A^L_\nu\star V^{L\mu\nu}- \\ -2A^L_\mu\star A^L_\nu\star A^{L\mu}\star A^{L\nu}+6A^L_\mu\star A^{L\mu}\star A^L_\nu\star A^{L\nu}\Big)$$

the part $\mathcal{A}^+_{\sigma_L}$ can be cancelled by a star-local counterterm. The other part, $\mathcal{A}^-_{\sigma_L}$, cannot be cancelled by a counterterm. This part is called the topological part of the anomaly. It is scheme-independent, and plays a more important role than $\mathcal{A}^+_{\sigma_L}$. Below we discuss the topological part only.

The anomaly related to the σ_R transformations can be obtained by applying the same rule as in section 2: one has to take (4.24) and (4.25), replace L by R everywhere, and invert the order of all multipliers. The topological part of the anomaly reads

$$\mathcal{A}^-_{\sigma_R} = \frac{-i}{48\pi^2}\int d^4x\sigma_R\star\epsilon^{\mu\nu\rho\sigma}\Big(3iV^R_{\mu\nu}\star V^R_{\rho\sigma}-iA^R_{\mu\nu}\star A^R_{\rho\sigma}+ \\ +2(V^R_{\mu\nu}\star A^R_\rho\star A^R_\sigma+A^R_\mu\star A^R_\nu\star V^R_{\rho\sigma})+8A^R_\mu\star V^R_{\nu\rho}\star A^R_\sigma+ \\ +4iA^R_\mu\star A^R_\nu\star A^R_\rho\star A^R_\sigma\Big)\,. \tag{4.26}$$

Again our results agree with previous calculations of the abelian anomalies without V^R-fields and axial vector fields [37 – 40]. We also like to mention a couple of recent publications which discuss the Fujikawa approach [41] and the operator approach [42] to the anomalies. Chiral anomaly on the noncommutative torus was calculated in [43].

If $\theta^{\mu\nu}$ is degenerate, the heat kernel coefficient a_n can contain mixed contributions. Consequently, non-planar contributions to the anomaly appear [40].

We conclude this section by constructing a model in four dimensions which has zero axial anomaly. Let us consider the action (4.1) where we choose

$$A^L_\mu = -A^R_\mu \equiv A_\mu \,, \qquad V^L_\mu = -V^R_\mu \equiv V_\mu \,. \tag{4.27}$$

Then also $V^L_{\mu\nu} = -V^R_{\mu\nu}$, $A^L_{\mu\nu} = -A^R_{\mu\nu}$, etc. This model has a trivial commutative limit. The relations (4.27) are preserved by gauge and axial gauge symmetries with the parameters restricted according to the relation

$$w_L = -w_R \,, \qquad \sigma_L = -\sigma_R \,. \tag{4.28}$$

Obviously, the (topological part of the) anomaly is automatically zero:

$$\mathcal{A}^-_{\sigma_L} + \mathcal{A}^-_{\sigma_R} = 0 \,. \tag{4.29}$$

Cancellations of gauge anomalies in noncommutative theories were discussed, e.g., in [44].

5. Conclusions

In this paper we constructed the heat kernel expansion for the operators which contain both left and right Moyal multiplications. We found two types of the terms. The terms of the first type are star-local and depend either on right or on left fields. The other terms are non-local, and they contain mixtures of left and right fields. Next we applied our results to the ϕ^4 theory and constructed a large mass and strong noncommutativity expansion of the effective action and of the vacuum energy. Then we calculated the axial anomaly, which do not contain mixed (non-planar) contributions and, in fact, looks rather standard. We also found a model where the topological part of the axial anomaly is identically zero.

Our work extends considerably the class of the operators on noncommutative spaces for which the heat kernel expansion is known. Possible applications of the results are not exhausted the the examples given above. It would be interesting to consider the consequences for the spectral action principle and to calculate quantum corrections to noncommutative instantons and solitons. As a more formal development one can consider a degenerate noncommutativity parameter (cf. [13]). In this case, some non-planar contributions to the anomalies should appear leading interesting physical consequences (cf. the discussion in [45, 40, 46]).

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